CSE 250
Data Structures

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Day 19
BFS, Shortest Path, Priority Queues

BFS

```
object VertexLabel extends Enumeration
  { val UNEXPLORED, VISITED = Value }

object EdgeLabel extends Enumeration
  { val UNEXPLORED, SPANNING, CROSS = Value }

def BFS(graph: Graph[VertexLabel.Value, EdgeLabel.Value]) {
  for(v <- graph.vertices) { v.setLabel(VertexLabel.UNEXPLORED) }
  for(e <- graph.edges)    { e.setLabel(EdgeLabel.UNEXPLORED) }
  for(v <- graph.vertices) {
    if(v.label == VertexLabel.UNEXPLORED){
      BFSOne(graph, v)
    }
  }
}
```

BFSOne

```
def BFSOne(graph: Graph[…], start: Graph[…]#Vertex) {
  val work = mutable.Queue[Graph[...]#Vertex]()
  work.enqueue(start)
  start.setLabel(VertexLabel.VISITED)
  while (!work.isEmpty) {
    v = work.dequeue()
    for(e <- v.incident) {
      if(e.label == EdgeLabel.UNEXPLORED){
        val w = e.getOpposite(v)
        if(w.label == VertexLabel.UNEXPLORED){
          work.enqueue(w)
          w.setLabel(VertexLabel.VISITED)
          e.setLabel(EdgeLabel.SPANNING)
        } else {
          e.setLabel(EdgeLabel.CROSS)
        }
      }
    }
  }
}
```

BFSOne

```
def BFSOne(graph: Graph[…], start: Graph[…]#Vertex) {
  val work = mutable.Queue[Graph[...]#Vertex]()
  work.enqueue(start)
  start.setLabel(VertexLabel.VISITED)
  while (!work.isEmpty) {
    v = work.dequeue()
    for(e <- v.incident) {
      if(e.label == EdgeLabel.UNEXPLORED){
        val w = e.getOpposite(v)
        if(w.label == VertexLabel.UNEXPLORED){
          work.enqueue(w)
          w.setLabel(VertexLabel.VISITED)
          e.setLabel(EdgeLabel.SPANNING)
        } else {
          e.setLabel(EdgeLabel.CROSS)
        }
      }
    }
  }
}
```

Create a work list of "nodes to visit", and add our start node

BFSOne

```
def BFSOne(graph: Graph[…], start: Graph[…]#Vertex) {
  val work = mutable.Queue[Graph[...]#Vertex]()
  work.enqueue(start)
  start.setLabel(VertexLabel.VISITED)
  while (!work.isEmpty) {
    v = work.dequeue()
    for(e <- v.incident) {
      if(e.label == EdgeLabel.UNEXPLORED){
        val w = e.getOpposite(v)
        if(w.label == VertexLabel.UNEXPLORED){
          work.enqueue(w)
          w.setLabel(VertexLabel.VISITED)
          e.setLabel(EdgeLabel.SPANNING)
        } else {
          e.setLabel(EdgeLabel.CROSS)
        }
      }
    }
  }
}
```

Loop as long as we have more work to do, dequeueing an item each iteration

BFSOne

```
def BFSOne(graph: Graph[…], start: Graph[…]#Vertex) {
  val work = mutable.Queue[Graph[...]#Vertex]()
  work.enqueue(start)
  start.setLabel(VertexLabel.VISITED)
  while (!work.isEmpty) {
    v = work.dequeue()
    for(e <- v.incident) {
      if(e.label == EdgeLabel.UNEXPLORED){
        val w = e.getOpposite(v)
        if(w.label == VertexLabel.UNEXPLORED){
          work.enqueue(w)
          w.setLabel(VertexLabel.VISITED)
          e.setLabel(EdgeLabel.SPANNING)
        } else {
          e.setLabel(EdgeLabel.CROSS)
        }
      }
    }
  }
}
```

If we find a new node, enqueue it to be explored, and label appropriately

BFSOne

```
def BFSOne(graph: Graph[…], start: Graph[…]#Vertex) {
  val work = mutable.Queue[Graph[...]#Vertex]()
  work.enqueue(start)
  start.setLabel(VertexLabel.VISITED)
  while (!work.isEmpty) {
    v = work.dequeue()
    for(e <- v.incident) {
      if(e.label == EdgeLabel.UNEXPLORED){
        val w = e.getOpposite(v)
        if(w.label == VertexLabel.UNEXPLORED){
          work.enqueue(w)
          w.setLabel(VertexLabel.VISITED)
          e.setLabel(EdgeLabel.SPANNING)
        } else {
          e.setLabel(EdgeLabel.CROSS)
        }
      }
    }
  }
}
```

…if we have already visited it, then don't add it to be explored

BFS - Complexity

```
def BFSOne(graph: Graph[…], start: Graph[…]#Vertex) {
  /* O(1) */
  while (!work.isEmpty) {
    /* O(1) */
    /* O(deg(v)) times */ {
      /* O(1) */{
        /* O(1) */
        /* O(1) */{
          /* O(1) */
          /* O(1) */
          /* O(1) */
        } else {
          /* O(1) */
        }
      }
    }
  }
}
```

BFS - Complexity

```
def BFSOne(graph: Graph[…], start: Graph[…]#Vertex) {
  /* O(1) */
  while (!work.isEmpty) {
    /* O(1) */
    /* O(deg(v)) times */ {
      /* O(1) */{
        /* O(1) */
        /* O(1) */{
          /* O(1) */
          /* O(1) */
          /* O(1) */
        } else {
          /* O(1) */
        }
      }
    }
  }
}
```

Each vertex is enqueued exactly once

BFS - Complexity

```
def BFSOne(graph: Graph[…], start: Graph[…]#Vertex) {
  /* O(1) */
  while (!work.isEmpty) {
    /* O(1) */
    /* O(deg(v)) times */ {
      /* O(1) */{
        /* O(1) */
        /* O(1) */{
          /* O(1) */
          /* O(1) */
          /* O(1) */
        } else {
          /* O(1) */
        }
      }
    }
  }
}
```

Each vertex is enqueued exactly once

The cost of each vertex, ***v***, is ***O*****(deg(*****v*****))**

Breadth-First Search Complexity

What is the sum over all iterations in **`BFSOne`**?

$$\sum_{v \in V} O(deg(v))$$

$$= O(\sum_{v \in V} deg(v))$$

$$= O(2|E|)$$

$$= O(|E|)$$

Breadth-First Search Complexity

In summary…

1. Mark the vertices `UNVISITED`

Breadth-First Search Complexity

In summary…

1. Mark the vertices `UNVISITED` $O(|V|)$

Breadth-First Search Complexity

In summary...

1. Mark the vertices `UNVISITED` $O(|V|)$
2. Mark the edges `UNVISITED`

Breadth-First Search Complexity

In summary...

1. Mark the vertices **`UNVISITED`** $O(|V|)$
2. Mark the edges **`UNVISITED`** $O(|E|)$

Breadth-First Search Complexity

In summary...

1. Mark the vertices **`UNVISITED`** $O(|V|)$
2. Mark the edges **`UNVISITED`** $O(|E|)$
3. Add each vertex to the work queue

Breadth-First Search Complexity

In summary...

1. Mark the vertices **`UNVISITED`** $O(|V|)$
2. Mark the edges **`UNVISITED`** $O(|E|)$
3. Add each vertex to the work queue $O(|V|)$

Breadth-First Search Complexity

In summary…

1. Mark the vertices **`UNVISITED`** $O(|V|)$
2. Mark the edges **`UNVISITED`** $O(|E|)$
3. Add each vertex to the work queue $O(|V|)$
4. Process each vertex

Breadth-First Search Complexity

In summary…

1.	Mark the vertices **`UNVISITED`**	$O(\|V\|)$
2.	Mark the edges **`UNVISITED`**	$O(\|E\|)$
3.	Add each vertex to the work queue	$O(\|V\|)$
4.	Process each vertex	$O(\|E\|)$

Breadth-First Search Complexity

In summary…

1. Mark the vertices **`UNVISITED`**	$O(\|V\|)$
2. Mark the edges **`UNVISITED`**	$O(\|E\|)$
3. Add each vertex to the work queue	$O(\|V\|)$
4. Process each vertex	$O(\|E\|)$
	$O(\|V\| + \|E\|)$

Queues vs Stacks

Thought Experiment: How is the use of a Queue related to traversal order?

Queues vs Stacks

Thought Experiment: How is the use of a Queue related to traversal order? What if we used a Stack instead?

BFSOne

```
def BFSOne(graph: Graph[…], start: Graph[…]#Vertex) {
  val work = mutable.Queue[Graph[...]#Vertex]()
  work.enqueue(start)
  start.setLabel(VertexLabel.VISITED)
  while (!work.isEmpty) {
    v = work.dequeue()
    for(e <- v.incident) {
      if(e.label == EdgeLabel.UNEXPLORED){
        val w = e.getOpposite(v)
        if(w.label == VertexLabel.UNEXPLORED){
          work.enqueue(w)
          w.setLabel(VertexLabel.VISITED)
          e.setLabel(EdgeLabel.SPANNING)
        } else {
          e.setLabel(EdgeLabel.CROSS)
        }
      }
    }
  }
}
```

DFSOneNoRecursion

```
def DFSOneNoRecursion(graph: Graph[…], start: Graph[…]#Vertex) {
  val work = mutable.Stack[Graph[...]#Vertex]()
  work.push(start)
  start.setLabel(VertexLabel.VISITED)
  while (!work.isEmpty) {
    v = work.pop()
    for(e <- v.incident) {
      if(e.label == EdgeLabel.UNEXPLORED){
        val w = e.getOpposite(v)
        if(w.label == VertexLabel.UNEXPLORED){
          work.push(w)
          w.setLabel(VertexLabel.VISITED)
          e.setLabel(EdgeLabel.SPANNING)
        } else {
          e.setLabel(EdgeLabel.BACK)
        }
      }
    }
  }
}
```

Queues vs Stacks

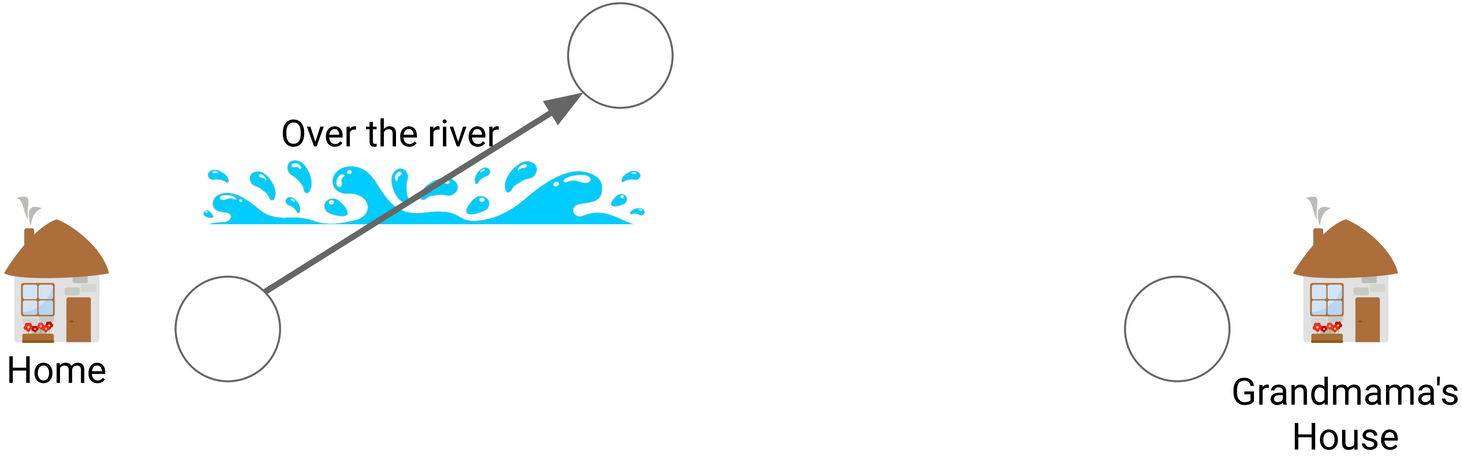
Observation: The recursive version of DFS was using a Stack all along…the call stack!

DFS Traversal vs BFS Traversal

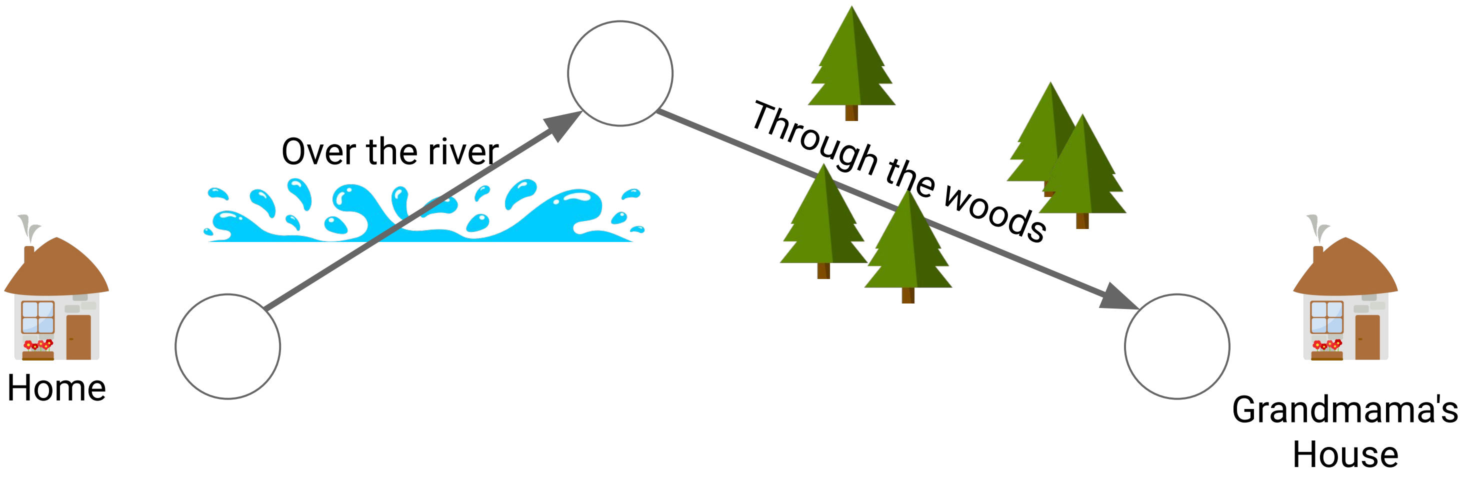
Application	DFS	BFS
Spanning Trees	✓	✓
Connected Components	✓	✓
Paths/Connectivity	✓	✓
Cycles	✓	✓
Shortest Paths		✓
Articulation Points	✓	

Shortest Paths

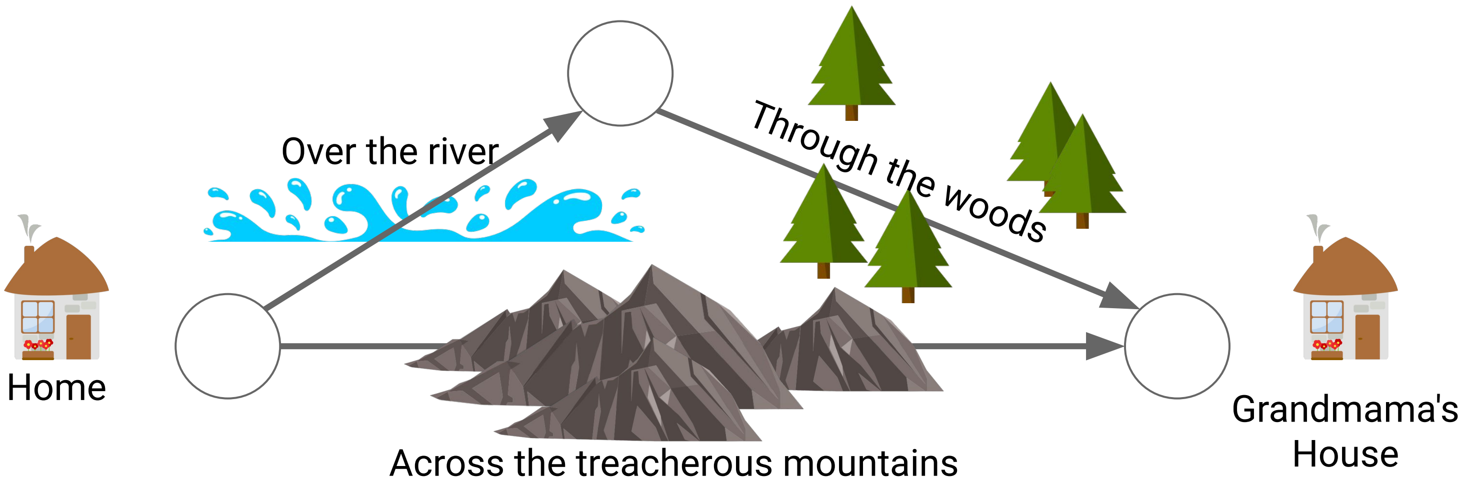
Shortest Paths



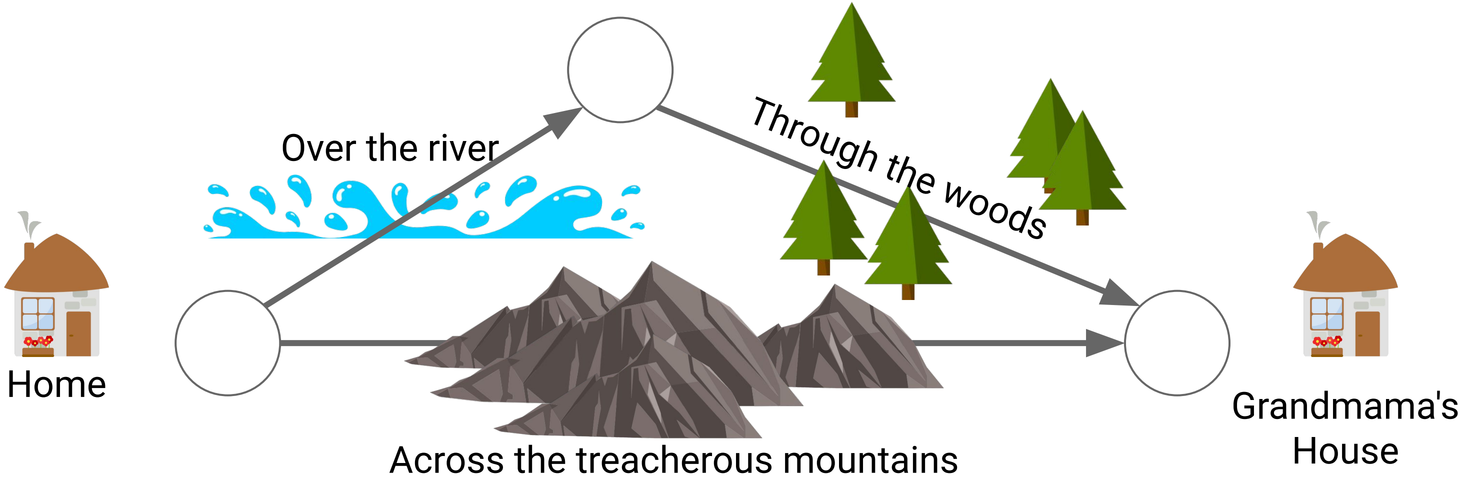
Shortest Paths



Shortest Paths

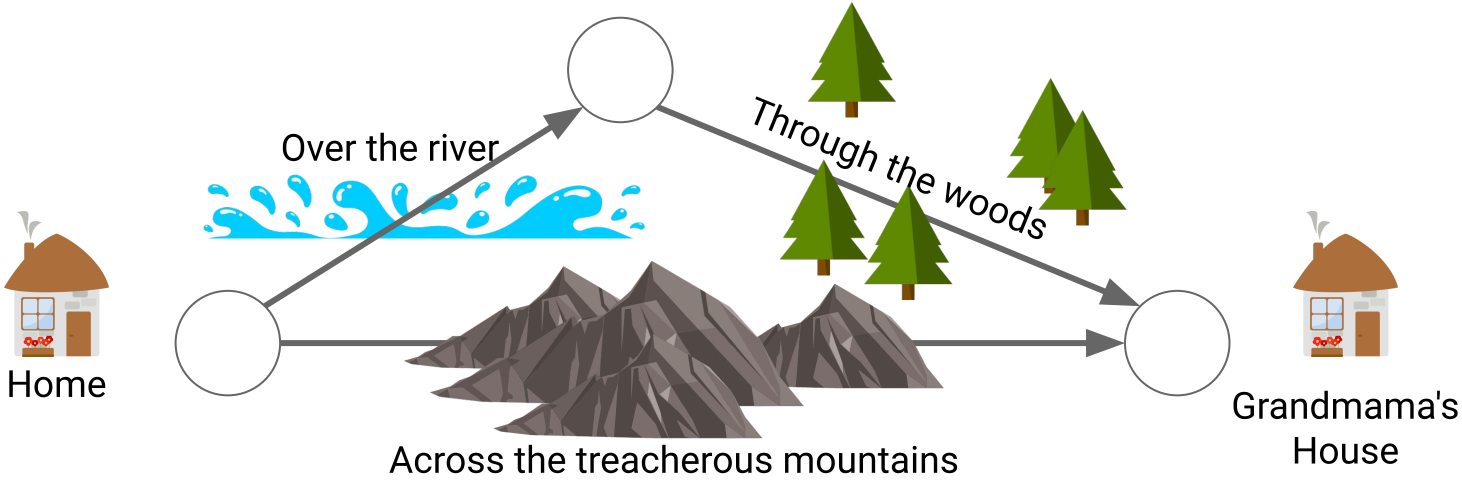


Shortest Paths



BFS will always find the path with the **fewest edges**…

Shortest Paths



BFS will always find the path with the **fewest edges**...

Not all edges in a real world graph are necessarily created equal!

Which path is actually the best/shortest?

Weighted Graphs

A **weighted graph** is a pair of:

- a graph $\boldsymbol{G = (V,E)}$
- A weight function $\boldsymbol{\omega(e)}$ that assigns a real number (called an **edge weight**) to each edge $\boldsymbol{e \in E}$

Examples of Weights:

- Latency of a network connection
- Distance between two cities
- Time between two metro stops
- Flow capacity between two points in a series of tubes

Shortest Path

Given:

- A weighted graph $\boldsymbol{G = (V,E,\omega)}$
- A start vertex ***start*** in $\boldsymbol{V}$
- An end vertex ***end*** in $\boldsymbol{V}$

Shortest Path

Given:

- A weighted graph $\boldsymbol{G = (V,E,\omega)}$
- A start vertex ***start*** in ***V***
- An end vertex ***end*** in ***V***

Goal:

- Produce a simple path ***P*** from ***start*** to ***end***…
- …that minimizes the sum of edge weights in the ***P***

BFSOne - Adding Level

```
def BFSOne(graph: Graph[…], start: Graph[…]#Vertex) {
  val work = mutable.Queue[Graph[...]#Vertex]()
  work.enqueue(start)
  start.setLabel(VertexLabel.VISITED)
  while (!work.isEmpty) {
    v = work.dequeue()
    for(e <- v.incident) {
      if(e.label == EdgeLabel.UNEXPLORED){
        val w = e.getOpposite(v)
        if(w.label == VertexLabel.UNEXPLORED){
          work.enqueue(w)
          w.setLabel(VertexLabel.VISITED)
          e.setLabel(EdgeLabel.SPANNING)
        } else {
          e.setLabel(EdgeLabel.CROSS)
        }
      }
    }
  }
}
```

BFSOne - Adding Level

```
def BFSOne(graph: Graph[…], start: Graph[…]#Vertex) {
  val work = mutable.Queue[Graph[...]#Vertex, Int]()
  work.enqueue((start,0))
  start.setLabel(VertexLabel.VISITED)
  while (!work.isEmpty) {
    (v, level) = work.dequeue()
    for(e <- v.incident) {
      if(e.label == EdgeLabel.UNEXPLORED){
        val w = e.getOpposite(v)
        if(w.label == VertexLabel.UNEXPLORED){
          work.enqueue((w, level + 1))
          w.setLabel(VertexLabel.VISITED)
          e.setLabel(EdgeLabel.SPANNING)
        } else {
          e.setLabel(EdgeLabel.CROSS)
        }
      }
    }
  }
}
```

BFSOne - Shortest Path

```
def BFSOne(graph: Graph[…], start: Graph[…]#Vertex) {
  val work = mutable.Queue[Graph[...]#Vertex, Int]()
  work.enqueue((start,0))
  start.setLabel(VertexLabel.VISITED)
  while (!work.isEmpty) {
    (v, level) = work.dequeue()
    for(e <- v.incident) {
      if(e.label == EdgeLabel.UNEXPLORED){
        val w = e.getOpposite(v)
        if(w.label == VertexLabel.UNEXPLORED){
          work.enqueue((w, level + 1))
          w.setLabel(VertexLabel.VISITED)
          e.setLabel(EdgeLabel.SPANNING)
        } else {
          e.setLabel(EdgeLabel.CROSS)
        }
      }
    }
  }
}
```

BFS always adds 1 to the level when exploring new nodes. **One** edge adds **one** to the level.

BFSOne - Shortest Path

```
def BFSOne(graph: Graph[…], start: Graph[…]#Vertex) {
  val work = mutable.Queue[Graph[...]#Vertex, Int]()
  work.enqueue((start,0))
  start.setLabel(VertexLabel.VISITED)
  while (!work.isEmpty) {
    (v, level) = work.dequeue()
    for(e <- v.incident) {
      if(e.label == EdgeLabel.UNEXPLORED){
        val w = e.getOpposite(v)
        if(w.label == VertexLabel.UNEXPLORED){
          work.enqueue((w, level + 1))
          w.setLabel(VertexLabel.VISITED)
          e.setLabel(EdgeLabel.SPANNING)
        } else {
          e.setLabel(EdgeLabel.CROSS)
        }
      }
    }
  }
}
```

Consequence: Dequeue reads vertices in ascending order of level. (FIFO)

BFSOne - Shortest Path

```
def BFSOne(graph: Graph[…], start: Graph[…]#Vertex) {
  val work = mutable.Queue[Graph[...]#Vertex, Int]()
  work.enqueue((start,0))
  start.setLabel(VertexLabel.VISITED)
  while (!work.isEmpty) {
    (v, level) = work.dequeue()
    for(e <- v.incident) {
      if(e.label == EdgeLabel.UNEXPLORED){
        val w = e.getOpposite(v)
        if(w.label == VertexLabel.UNEXPLORED){
          work.enqueue((w, level + 1))
          w.setLabel(VertexLabel.VISITED)
          e.setLabel(EdgeLabel.SPANNING)
        } else {
          e.setLabel(EdgeLabel.CROSS)
        }
      }
    }
  }
}
```

Consequence: Dequeue reads vertices in ascending order of level. (FIFO)

Therefore: The first time we reach a vertex, it is via the fewest number of edges from start.

BFS and Shortest Path

Observation: Breadth-First Search finds paths with the fewest number of edges. This is equivalent to finding the shortest path with $\boldsymbol{\omega(e) = 1}$ for all $\boldsymbol{e}$

What changes if we allow $\boldsymbol{\omega(e)}$ *to vary?*

Detailed Example

UNEXPLORED

START

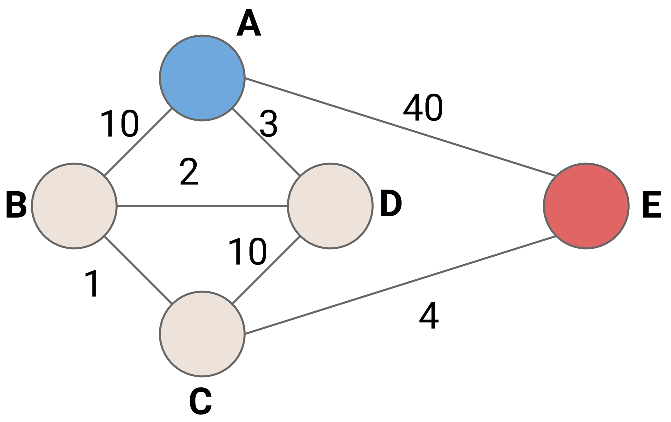
TARGET

VISITED

UNEXPLORED

SPANNING

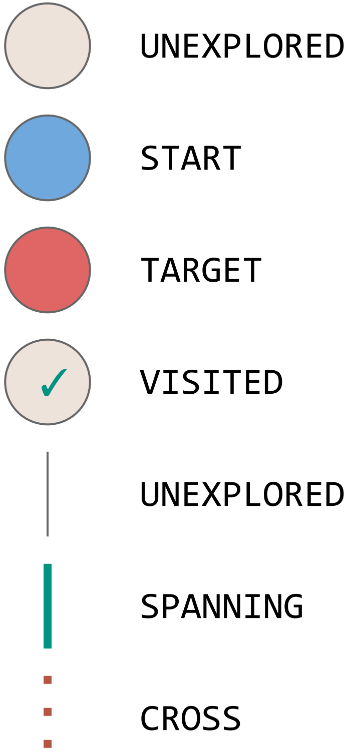
CROSS

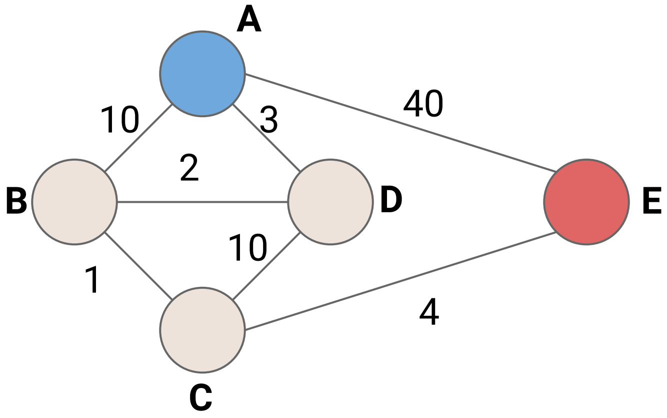


Call Stack

Work Queue

Detailed Example



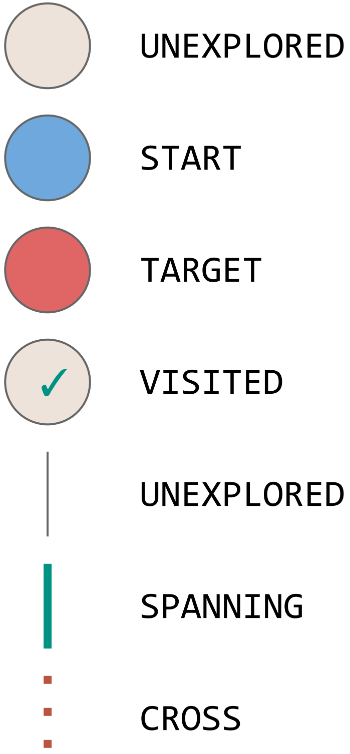


Call Stack

BFS(G)

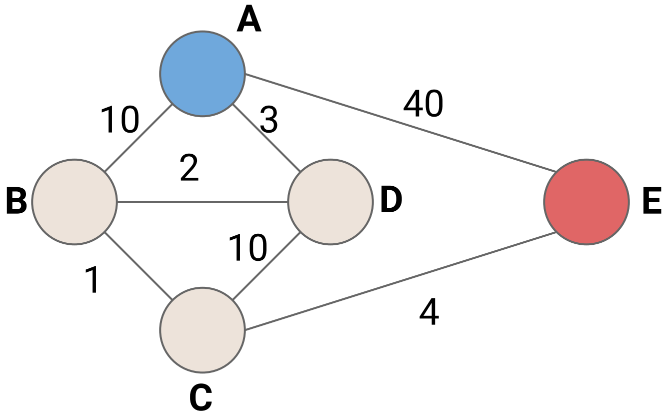
Work Queue

Detailed Example



<u>Call Stack</u>
BFS(G)
BFSOne(G,A)

<u>Work Queue</u>



Detailed Example

UNEXPLORED

START

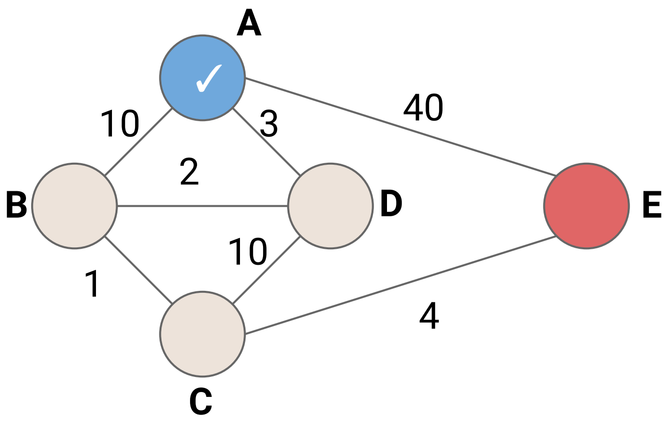
TARGET

VISITED

UNEXPLORED

SPANNING

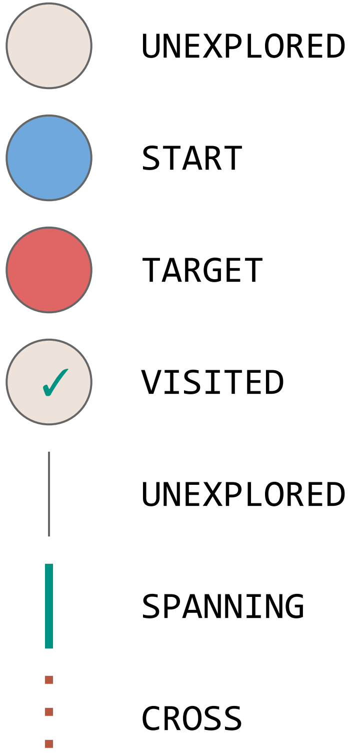
CROSS

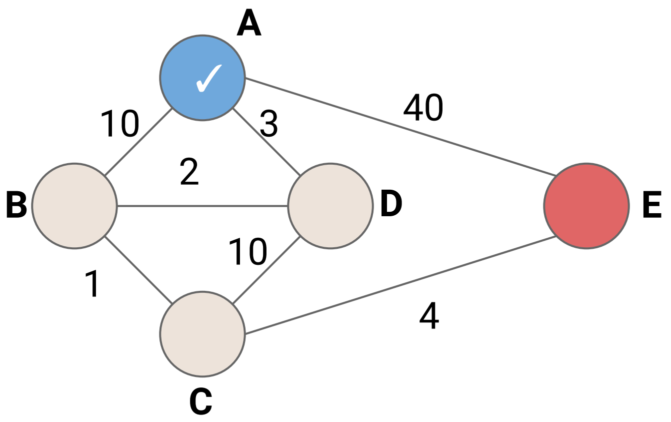


Call Stack
BFS(G)
BFSOne(G,A)

Work Queue
A

Detailed Example

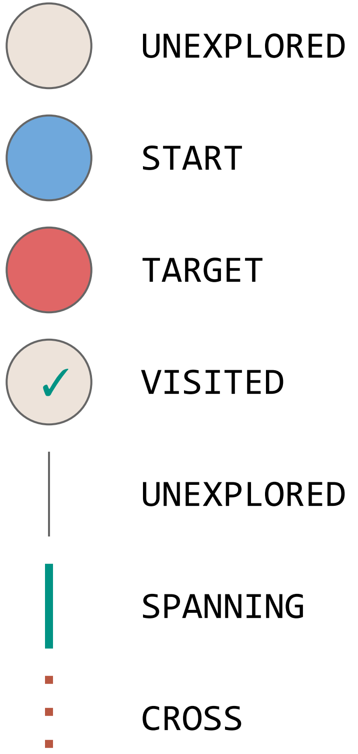


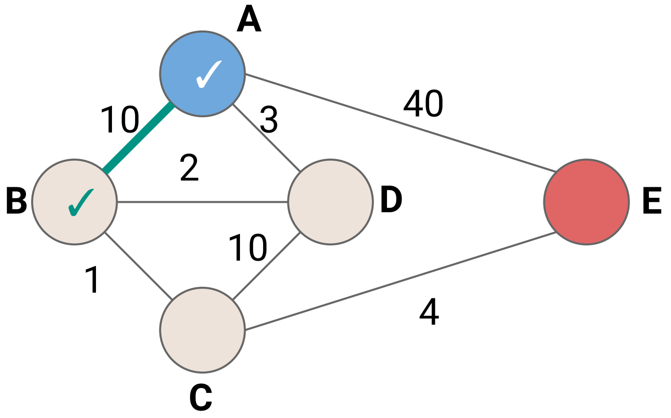


Call Stack
BFS(G)
BFSOne(G,A)

Work Queue
~~A~~

Detailed Example





<u>Call Stack</u>

BFS(G)

BFSOne(G,A)

<u>Work Queue</u>

~~**A**~~

B

Detailed Example

UNEXPLORED

START

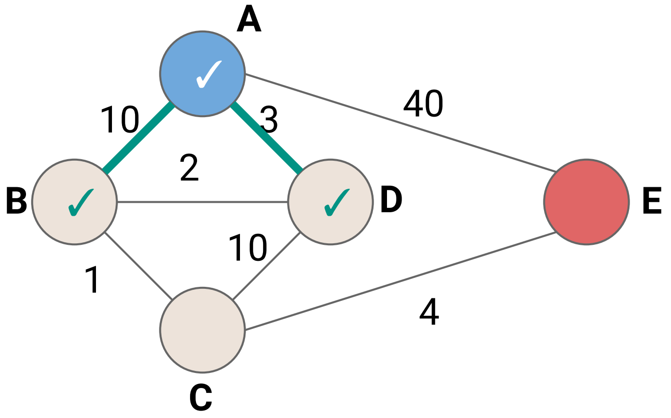
TARGET

VISITED

UNEXPLORED

SPANNING

CROSS



Call Stack

BFS(G)

BFSOne(G,A)

Work Queue

~~A~~

B

D

Detailed Example

UNEXPLORED

START

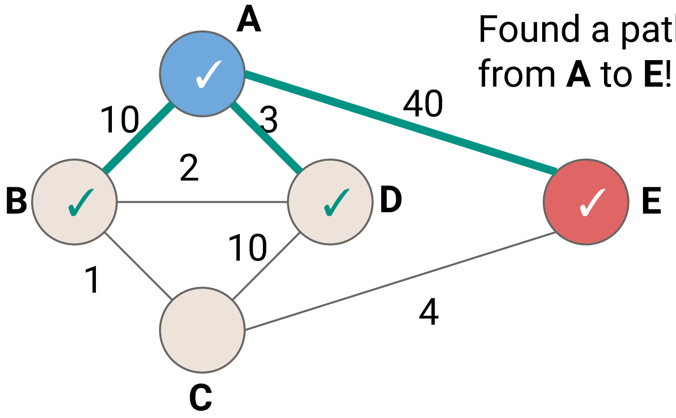
TARGET

VISITED

UNEXPLORED

SPANNING

CROSS



Found a path from **A** to **E**!

Call Stack

`BFS(G)`

`BFSOne(G,A)`

Work Queue

~~**A**~~

B

D

E

Detailed Example - Desired Path

UNEXPLORED

START

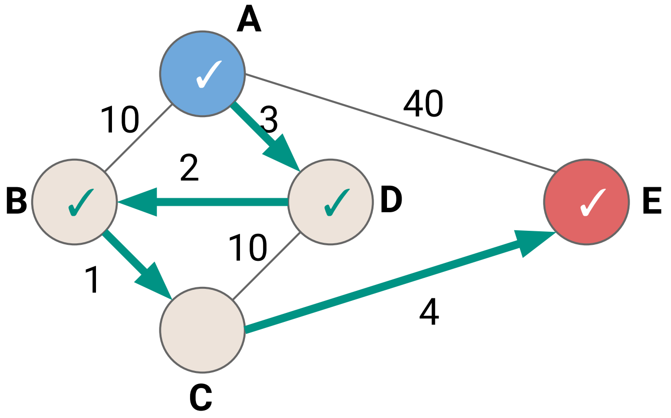
TARGET

VISITED

UNEXPLORED

SPANNING

CROSS



How do we find this path?

Shortest Path

Thought Experiment: How can we find the shortest path when not all edges are created equal?

Shortest Path

Thought Experiment: How can we find the shortest path when not all edges are created equal?

At any given point, what vertex should we explore next?

Attempt #1: Explore Smallest Edge

UNEXPLORED

START

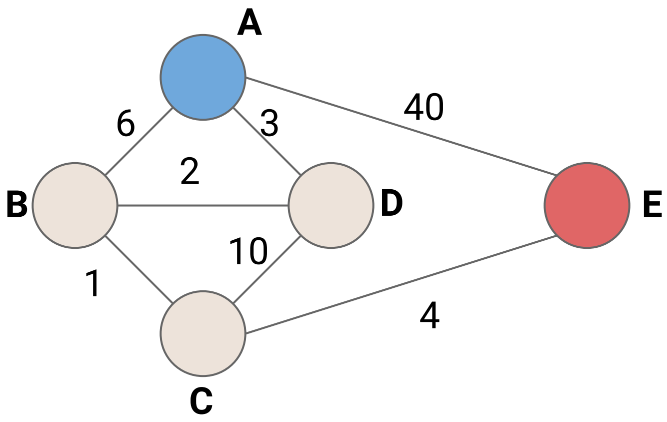
TARGET

✓ VISITED

UNEXPLORED

SPANNING

CROSS



Attempt #1: Explore Smallest Edge

UNEXPLORED

START

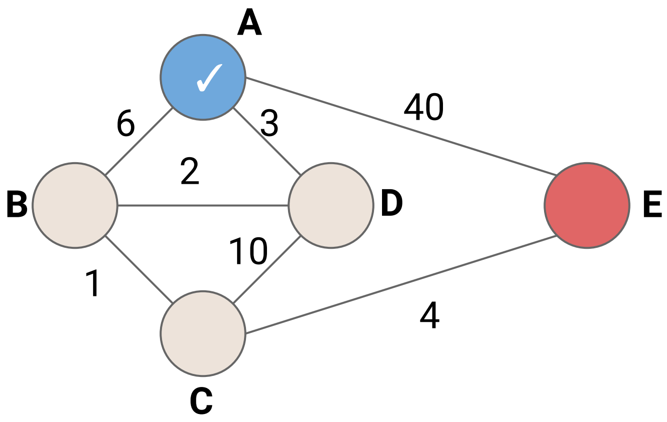
TARGET

VISITED

UNEXPLORED

SPANNING

CROSS



Attempt #1: Explore Smallest Edge

UNEXPLORED

START

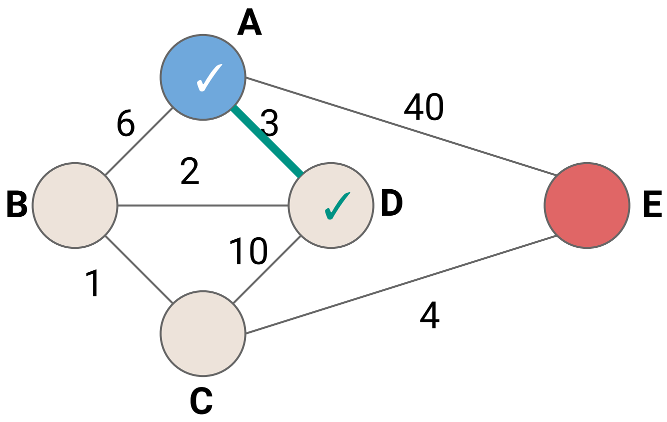
TARGET

VISITED

UNEXPLORED

SPANNING

CROSS



Attempt #1: Explore Smallest Edge

UNEXPLORED

START

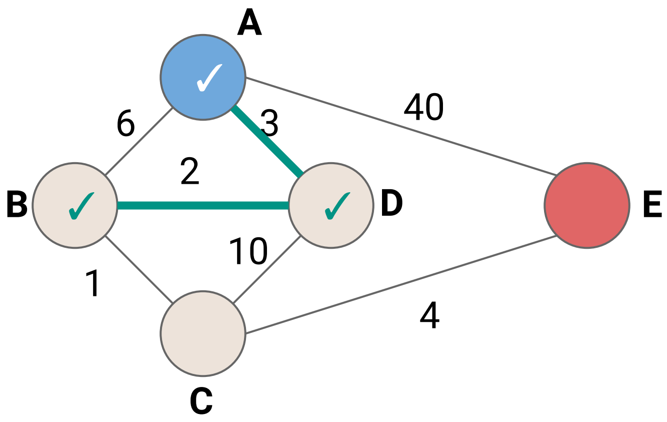
TARGET

VISITED

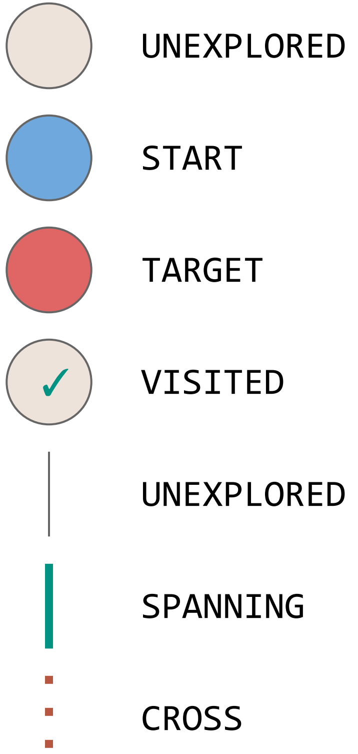
UNEXPLORED

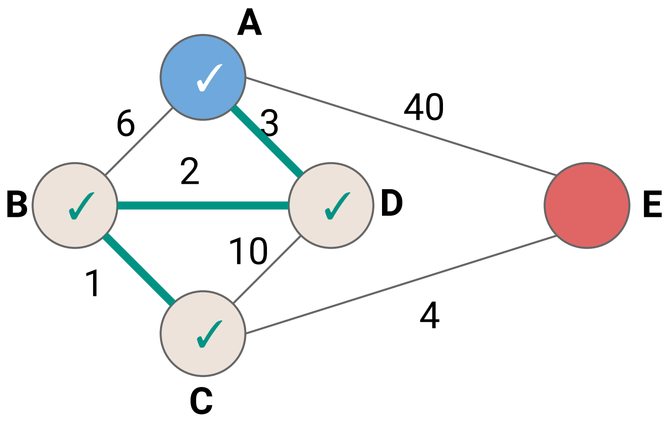
SPANNING

CROSS



Attempt #1: Explore Smallest Edge





Attempt #1: Explore Smallest Edge

UNEXPLORED

START

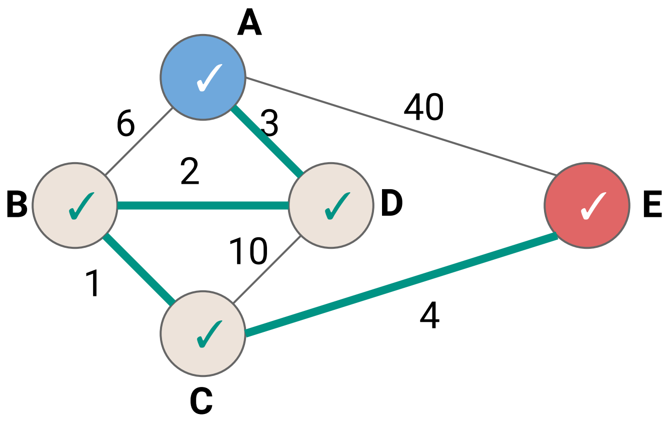
TARGET

VISITED

UNEXPLORED

SPANNING

CROSS



Attempt #1: Explore Smallest Edge

UNEXPLORED

START

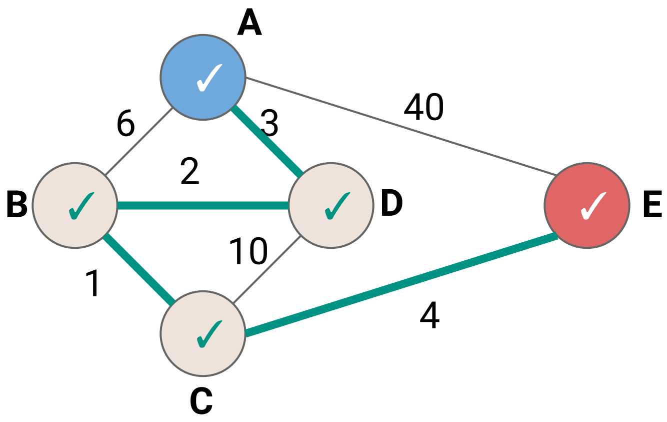
TARGET

VISITED

UNEXPLORED

SPANNING

CROSS



Attempt #1: Explore Smallest Edge

Will exploring the smallest available edge always work?

Desired Exploration Order

UNEXPLORED

START

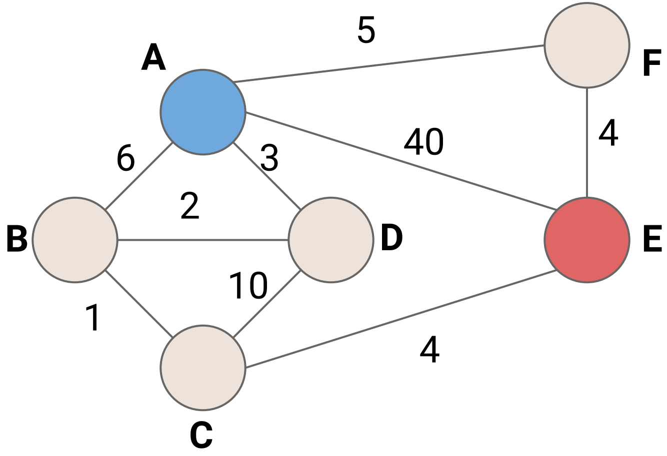
TARGET

✓ VISITED

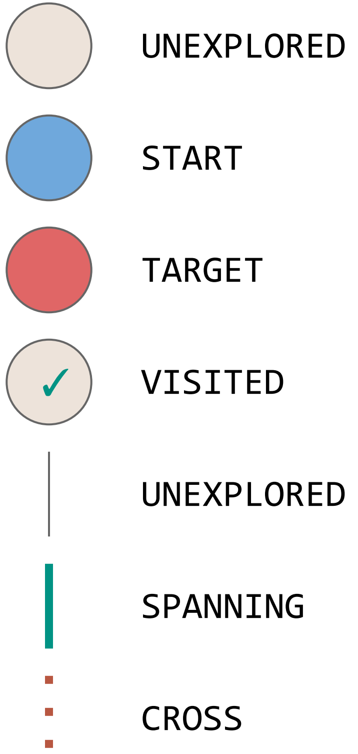
UNEXPLORED

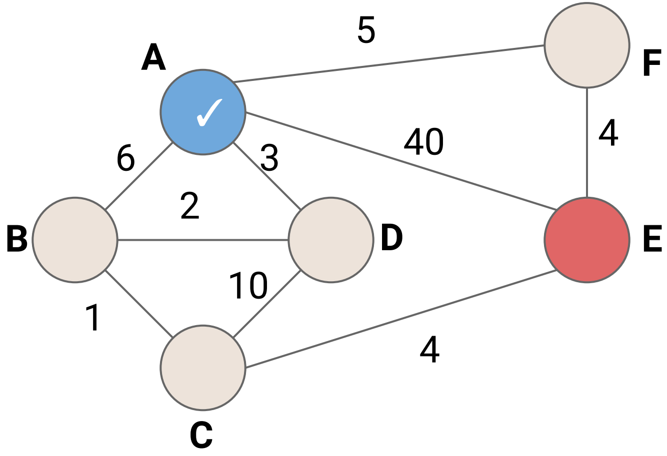
SPANNING

CROSS



Desired Exploration Order





Desired Exploration Order

UNEXPLORED

START

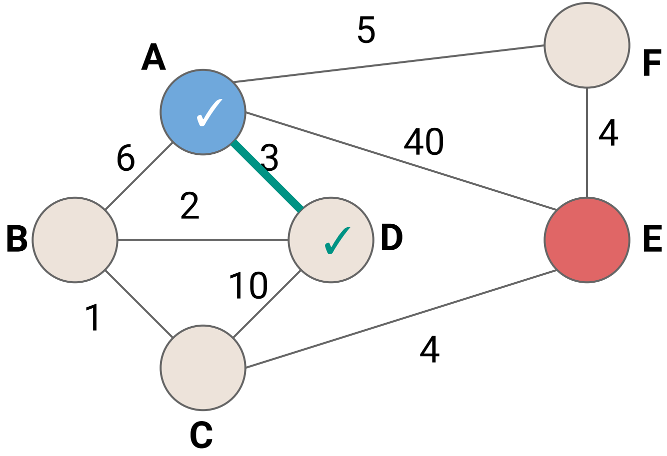
TARGET

VISITED

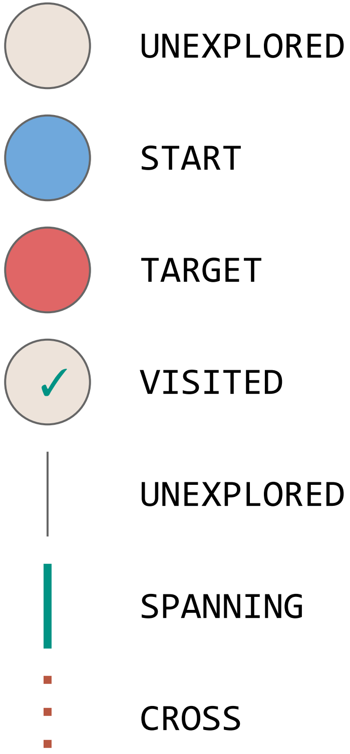
UNEXPLORED

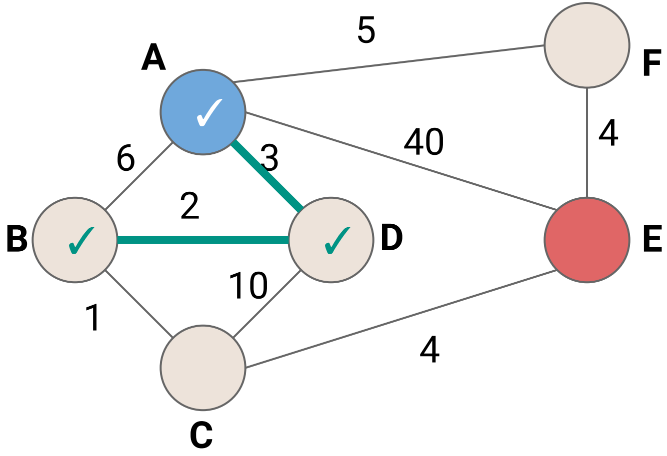
SPANNING

CROSS



Desired Exploration Order





Desired Exploration Order - Closest Vertex

UNEXPLORED

START

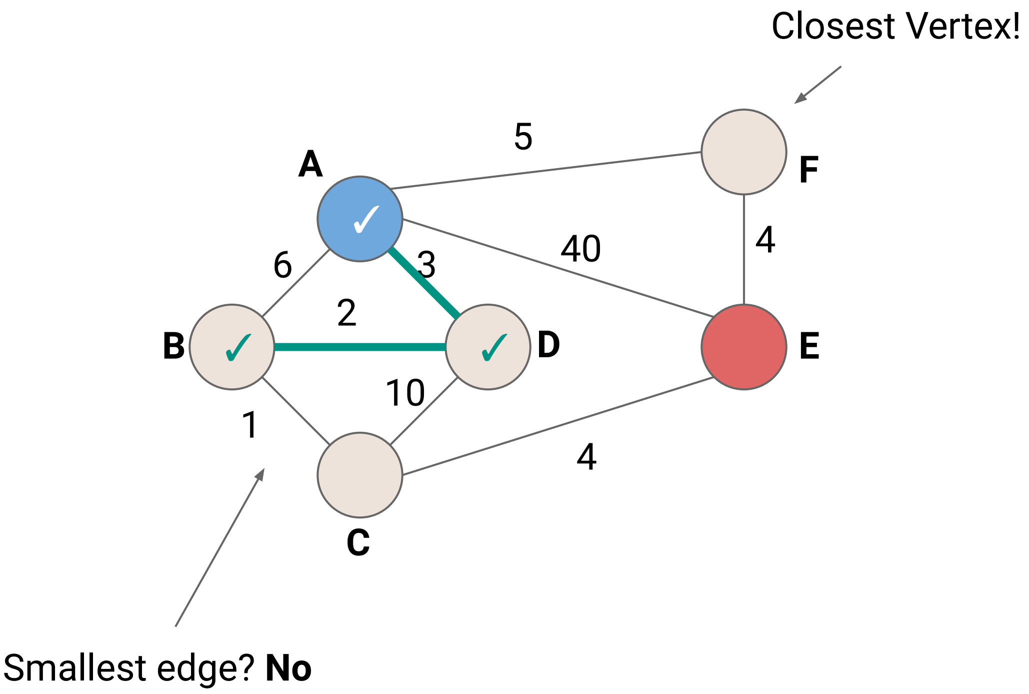
TARGET

VISITED

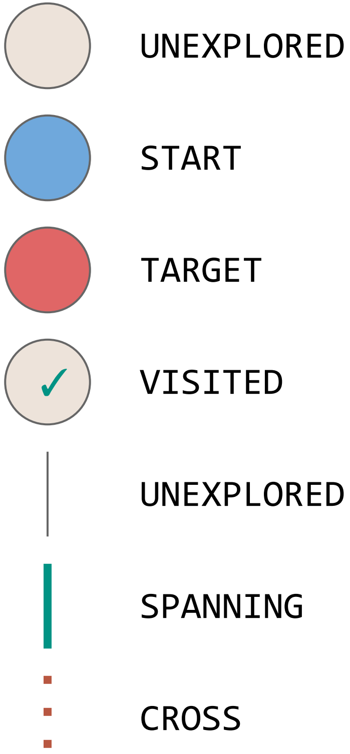
UNEXPLORED

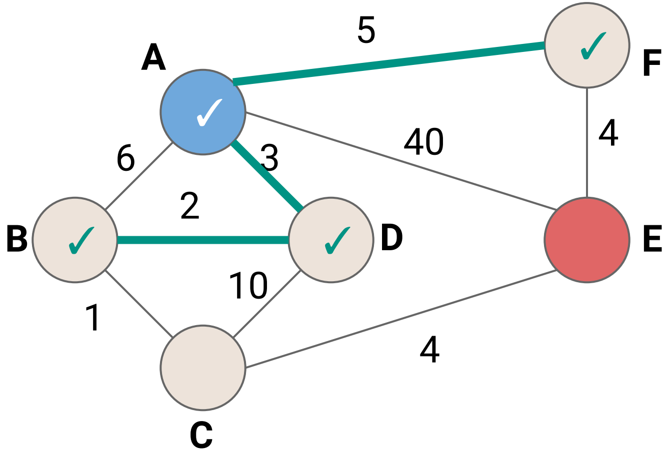
SPANNING

CROSS

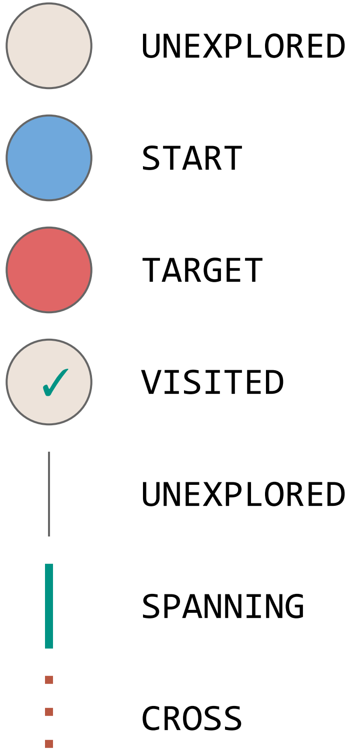


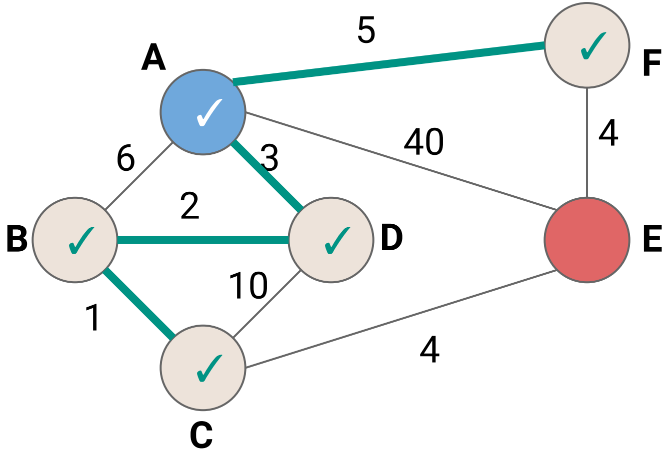
Desired Exploration Order



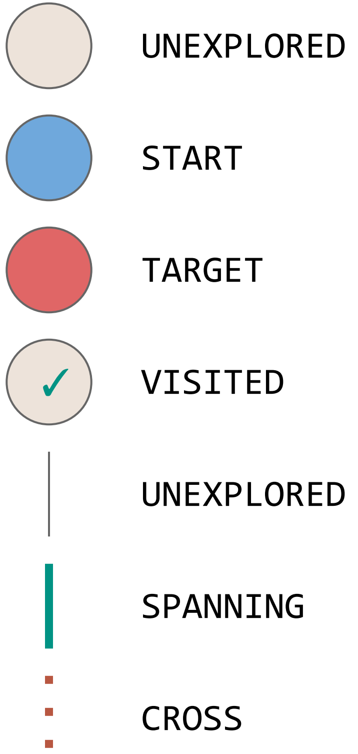


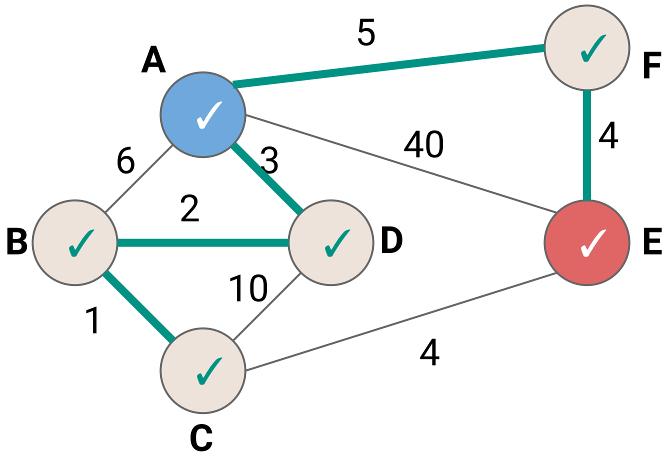
Desired Exploration Order





Desired Exploration Order





Path Found!

BFSOne - Shortest Path

```
def BFSOne(graph: Graph[…], start: Graph[…]#Vertex) {
  val work = mutable.Queue[Graph[...]#Vertex, Int]()
  work.enqueue((start,0))
  start.setLabel(VertexLabel.VISITED)
  while (!work.isEmpty) {
    (v, level) = work.dequeue()
    for(e <- v.incident) {
      if(e.label == EdgeLabel.UNEXPLORED){
        val w = e.getOpposite(v)
        if(w.label == VertexLabel.UNEXPLORED){
          work.enqueue((w, level + ω(e)))
          w.setLabel(VertexLabel.VISITED)
          e.setLabel(EdgeLabel.SPANNING)
        } else {
          e.setLabel(EdgeLabel.CROSS)
        }
      }
    }
  }
}
```

We want to be able to dequeue vertices in ascending order of distance…but how?

A New ADT...`PriorityQueue`

`PriorityQueue[A:Ordering]`

`enqueue(v: A)`

Insert value ***v*** into the priority queue

`head: A`

Peek at the highest priority value in the priority queue

`dequeue: A`

Remove the highest priority value in the priority queue

A New ADT...`PriorityQueue`

`PriorityQueue[A:Ordering]`

`enqueue(v: A)`

Insert value ***v*** into the priority queue

`head: A`

Peek at the **highest priority** value in the priority queue

`dequeue: A`

Remove the **highest priority** value in the priority queue